Vortex-line solitons in a periodically modulated Bose gas

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We study the nonlinear excitations of a vortex line in a Bose-Einstein condensate trapped in a one-dimensional optical lattice. We find that the classical Euler dynamics of the vortex results in a description of the vortex line in terms of a (discrete) one-dimensional Gross-Pitaevskii equation, which allows for both bright and gray soliton solutions. We discuss these solutions in detail and predict that it is possible to create vortex-line solitons with current experimental capabilities.

Introduction.— Solitons are special solutions of nonlinear differential equations with particle-like properties. In a Bose-Einstein condensate with a repulsive interaction a gray soliton corresponds to a density minimum for which the loss in interaction energy is balanced by the increase in the kinetic energy. Gray solitons in a Bose-Einstein condensate have been observed experimentally by imprinting a π phase step onto the condensate wave function [1, 2]. While a Bose-Einstein condensate with a repulsive interaction is stable against collapse, a condensate with attractive interactions may collapse. This collapse in first instance results in ever increasing density gradients. However, at later stages of the collapse the attractive interaction energy may be balanced by the kinetic energy, which tends to spread out the condensate wave function. Therefore, in a one-dimensional condensate a collapse leads to a creation of a train of bright solitons. Bright solitons in a Bose-Einstein condensate have indeed been observed experimentally [3, 4]. Also theoretically the collapse of the Bose-Einstein condensate into a train of bright solitons has been studied [5, 6].

In this Letter we point out the possibility of vortex-line solitons in a Bose-Einstein condensate in a one-dimensional optical lattice. As is well known, the physics of a dilute Bose gas in an optical lattice is described by the Bose-Hubbard model [7, 8], which in particular predicts a quantum phase transition from a superfluid state into a Mott-insulator state when the lattice becomes sufficiently deep. This transition was recently observed by Greiner *et al.* [9]. In the situation of interest to us here the one-dimensional optical lattice splits the Bose-Einstein condensate into a stack of weakly-interacting pancake condensates, and the vortex line intersects each one of them. We show that due to the inhomogeneous density distribution of the Bose-Einstein condensate, the dynamics of the vortex line is governed by a (discrete) one-dimensional nonlinear Schrödinger or Gross-Pitaevskii equation with the familiar soliton solutions.

These solitons are possible when the pressure due to the kinetic energy balances the pressure due to the interaction energy. In principle, these competing processes also exist for a vortex line in an ordinary magnetically trapped Bose-Einstein condensate without an optical lattice. In particular, the inhomogeneous condensate-density distribution in the radial direction gives rise to an interaction-like energy term and the energetic cost for bending the vortex line gives rise to the analog of the kinetic energy. However, in an ordinary Bose-Einstein condensate the vortex stiffness is orders of magnitude larger than in an optical lattice, where the vortex stiffness can be reduced by simply increasing the lattice depth. In practice this implies that the size of the soliton in an ordinary condensate is very large and, in particular, larger than the typical condensate size. Setting aside other complications this fact alone suggests that with current experimental capabilities vortex-line solitons are a possibility that only exists for a Bose-Einstein condensate in an optical lattice.

So-called optical vortex solitons [10] and solitonic vortices in a Bose-Einstein condensate [11] have been discussed before. Neither one of them is, however, related to the vortex-line solitons we discuss here. The former are related to the non-spreading propagation of a light beam with a vortex phase pattern through a nonlinear medium, while the latter refers to the soliton-like properties of a vortex moving through a very narrow channel. Moreover, in classical fluid dynamics Hashimoto [12] mapped the vortex line dynamics, in the so-called local induction approximation, into a nonlinear Schrödinger equation. As a result, solitons predicted by this formulation are only mathematically related to our vortex-line solitons and the physics is fundamentally different.

Theory.— We assume a Bose-Einstein condensate experiencing a one-dimensional optical lattice potential in the longitudinal direction. This lattice potential splits the Bose-Einstein condensate in N_s sites and each site has N atoms. In the radial direction the atoms are magnetically trapped by a harmonic trap with a trapping frequency ω_r. The magnetic trapping in the longitudinal direction is assumed to be so weak that it can be safely neglected. While the lattice is taken to be deep enough to allow us to use a tight-binding approximation and to include only the weak nearest-neighbor Josephson coupling, it is also taken to be shallow enough to support a superfluid state as opposed to the Mott-insulator

state [13].

In our earlier work [14, 15] we showed that up to second order in the vortex displacements from the center of the Bose-Einstein condensate the Hamiltonian for the vortex line is that of a set of coupled harmonic oscillators. The eigenmodes of this system are the Kelvin modes. These modes have been recently observed in a cigar shaped Bose-Einstein condensate in the absence of an optical lattice [16, 17]. However, if the vortex energy functional is expanded up to fourth order in the displacements, the physics of the vortex line turns out to be described by a one-dimensional Bose-Hubbard model. The corresponding Hamiltonian is given by

$$\begin{aligned}\hat{H} &= \sum_n \left[\hbar\omega_0 + J_V\right] \hat{v}_n^\dagger \hat{v}_n - \frac{J_V}{2} \sum_{\langle n,m\rangle} \hat{v}_m^\dagger \hat{v}_n \\ &+ \frac{V_0}{2} \sum_n \hat{v}_n^\dagger \hat{v}_n^\dagger \hat{v}_n \hat{v}_n, \end{aligned} \quad (1)$$

where $\langle n,m\rangle$ denotes the nearest-neighbor layers and the Kelvon operators $\hat{v}_n$ were defined in terms of the vortex displacements (x_n, y_n) as $\hat{x}_n = (R/2\sqrt{N})(\hat{v}_n^\dagger + \hat{v}_n)$ and $\hat{y}_n = (iR/2\sqrt{N})(\hat{v}_n^\dagger - \hat{v}_n)$, where R is the radial size of the pancake condensate.

In view of the complicated dynamics of the three-dimensional vortex line in a bulk superfluid, this relatively simple result is quite remarkable. Furthermore, as mentioned previously the Bose-Hubbard model shows a quantum phase transition from a superfluid into a Mott-insulator state when J_V/V_0 becomes sufficiently small. In our case this transition describes the break-up of the vortex line into individual pancake vortices due to quantum fluctuations. However, in the following we work solely in te superfluid regime, where a classical mean-field description is appropriate.

Using a Gaussian variational ansatz for the condensate wave function in each site [14] that is proportional to $\exp\left[-\left(x^2+y^2\right)/2R^2\right]$, we can analytically calculate the strength of the nearest-neighbor coupling to be

$$J_V = \frac{\hbar\omega_r}{4\pi^2}\Gamma\left[0, \frac{l_r^4}{R^4}\right]\left(\frac{\omega_L\lambda}{\omega_r l_r}\right)^2\left(\frac{\pi^2}{4}-1\right)\exp\left(-\frac{\lambda^2 m\omega_L}{4\hbar}\right)$$

and the interaction strength

$$V_0 = \frac{2\hbar\omega_r\left(l_r/R\right)^2\Gamma\left[0, l_r^4/R^4\right] - 3\hbar\omega_r\left(l_r/R\right)^2 - 4\hbar\Omega}{4N},$$

where $\omega_L = \sqrt{8\pi^2 V_L/m\lambda^2}$ is the oscillator frequency of the optical lattice, $l_r = \sqrt{\hbar/m\omega_r}$, $\Gamma[a,z]$ is the incomplete gamma function, $V_L\cos^2(2\pi z/\lambda)$ is the lattice potential, λ is the wavelength of the laser light, and Ω is the rotation frequency of the magnetic trap. Furthermore, a straight and slightly displaced vortex precesses around the condensate center with the frequency $\omega_0 = \left(\omega_r l_r^2/2R^2\right)\left(1-\Gamma\left[0, l_r^4/R^4\right]\right)+\Omega$. The radial size R of the Bose-Einstein condensate is determined by minimizing the condensate energy with the vortex line in the center of the condensate [15].

In Ref. [**?**] we discussed the equilibrium properties of a straight vortex when the interaction term is attractive and pointed out that in this case the quantum mechanical state of the vortex can become strongly squeezed. However, here make the usual classical approximation and study the nonlinear excitations of the vortex line. The solitonic solutions that we find can in principle also be squeezed under the appropriate conditions, but this interesting topic is deferred to future work.

Under the assumption of a coherent state for the Bose-Einstein condensate, the Euler dynamics of the vortex line in an optical lattice is given by the discrete Gross-Pitaevskii equation

$$\begin{aligned} i\dot{v}_n(t) &= -\frac{J_V}{2}\left[v_{n-1}(t) - 2v_n(t) + v_{n+1}(t)\right] \\ &+ \left[\omega_0 + V_0|v_n(t)|^2\right]v_n(t), \end{aligned} \quad (2)$$

where from now on we use units such that $\hbar = 1$. If the variation of the vortex positions is small over one lattice spacing this equation is simply the discretized version of the continuum Gross-Pitaevskii equation

$$i\dot{v}_n(t) = -\frac{J_V}{2}\frac{\partial^2 v_n(t)}{\partial n^2} + \left[\omega_0 + V_0|v_n(t)|^2\right]v_n(t). \quad (3)$$

As indicated by the above equation we from now on also choose to work with a longitudinal unit of length that equals the lattice spacing $\lambda/2$.

Dark vortex-line soliton.— Dark or gray soliton solutions of Eq. (3) are possible when the interaction is repulsive, i.e., $V_0 > 0$. This is the case in the non-rotating trap, for example. Explicitly the gray soliton solution moving with velocity v is given by

$$v_n(t) = \sqrt{n_0}\left\{i\cos\theta + \sin\theta\tanh\left[\sin\theta\frac{(n-vt)}{\xi}\right]\right\}e^{-i\mu t},$$

where $n_0 = \lim_{n\to\infty}|v_n|^2$, $\cos\theta = v/c$, $c = \sqrt{J_V V_0 n_0}$ is the speed of sound, $\mu = \omega_0 + V_0 n_0$, and the soliton size is $\xi = \sqrt{J_V/V_0 n_0}$ lattice spacings. While the gray soliton solution is simple to construct theoretically, its experimental preparation is challenging since it requires a good control over the position of the vortex in each site. One possible way to create a dark soliton is to prepare a vortex line that is tilted with respect to the optical potential. If the vortex line crosses the center of the condensate somewhere along the lattice, this initial state provides a prototype for the dark soliton. Other initial states can similarly resemble gray solitons. Unfortunately, the ensuing time evolution, which we have studied numerically, typically also generates a fair number of other vortex-line excitations, such as Kelvin modes and phonons. This makes the experimental observation complicated.

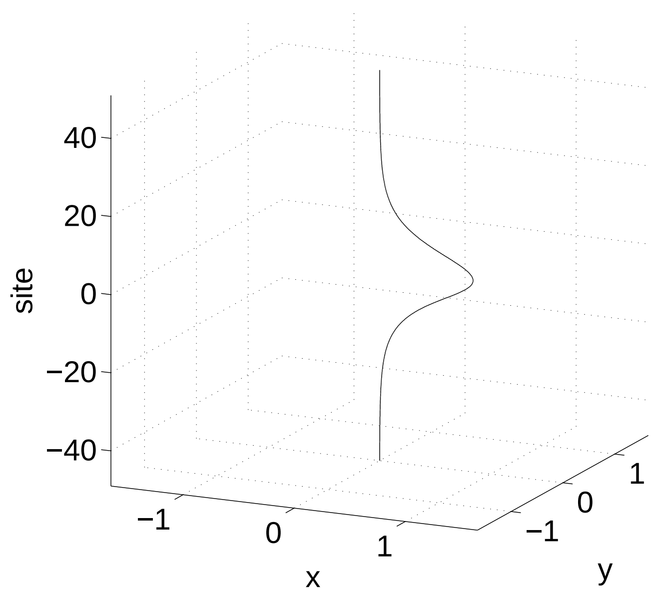

FIG. 1: A bright soliton solution of Eq. (4) with $N_0 = 1000$, $k = 0$, $J_V = 0.4\,\hbar\omega_r$, and $V_0 = -7\cdot 10^{-5}\,\hbar\omega_r$. These values for J_V and V_0 are realistic in an optical potential with a lattice spacing $d = 398\,\mathrm{nm}$, depth $V_L = 15\,E_r$, with 1000 ^{87}Rb atoms in each site, and a rotation frequency $\Omega = 0.2\,\omega_r$. here $\omega_r = 2\pi\cdot 100\,\mathrm{Hz}$ is the radial trapping frequency, $E_r = h^2/2m\lambda^2$ is the recoil energy, and the unit of length in the transverse direction is $\sqrt{\hbar/m\omega_r}$.

Bright vortex-line soliton.— Since experimental preparation of the dark soliton is complicated, we focus now on the regime where the interaction is attractive, i.e, $V_0 < 0$. This occurs, when the magnetic trap is rotated so fast that the vortex is energetically stable. With an attractive interaction there exists a solution of Eq. (3) corresponding to the bright vortex-line soliton moving along the lattice, which is given by

$$v_n(t) = \sqrt{\frac{N_0}{2\xi}}\frac{e^{-i\mu t+ikn}}{\cosh\left[\left(n - J_V kt/2\right)/\xi\right]}, \tag{4}$$

where k is the wavevector, $\mu = \omega_0 - J_V/2\xi^2 + J_V k^2/2$, and the size of the soliton is $\xi = 2J_V/|V_0|N$ lattice spacings. This solution is normalized such that $\int dn|v_n|^2 = N_0$. When $k = 0$, this solution is equivalent with a shape preserving precession of the curved vortex line. Since the chemical potential μ of the stationary soliton is smaller than ω_0, the soliton precesses around the center of the condensate with a lower precession frequency than a straight vortex line. In Fig. 1 we show an example of the bright vortex soliton with typical parameter values.

Soliton creation.— In order to gain quantitative understanding of the vortex-line collapse into a train of such solitons, we need to study the excitations around the homogeneous solution. At long wavelengths the Bogoliubov dispersion for a displaced but straight infinitely long vortex line corresponding to the density $|v_n|^2 = n_0$ is given by

$$\epsilon_k = J_V\sqrt{\frac{k^2}{2}\left(\frac{k^2}{2} + \frac{2V_0 n_0}{J_V}\right)}. \tag{5}$$

At wavelengths that are comparable to the lattice spacing the lattice discreteness becomes apparent and the dispersion is modified by the replacement $J_V k^2/2 \rightarrow J_V\left[1 - \cos(k)\right]$. However, for our purposes the long wavelength limit of the dispersion is sufficient. When $V_0 < 0$ the dispersion has imaginary energies and the system is dynamically unstable. The most unstable mode has the wavevector $k_0 = \sqrt{2|V_0|n_0/J_V}$ and the absolute value of its energy can be used to estimate the characteristic timescale τ for the collapse by $\tau = 1/|\epsilon_{k_0}| = 1/|V_0|n_0$. In contrast to the nearest-neighbor coupling strength J_V, V_0 is not very sensitive to the depth of the lattice potential. Therefore, the timescale for the vortex-line collapse is hard to tune by changing the lattice depth. A better way to tune the collapse time is to vary the initial vortex displacement. We demonstrate this in Fig. 2, which shows the behavior of τ with some typical parameter values as a function of the initial vortex-line displacement. It is clear from this figure that by displacing the vortex line sufficiently, the collapse can occur so quickly that the dissipative processes are unlikely to affect the vortex dynamics.

As the wavevector $k_0 = 2\pi/\lambda_0$ indicates the wavelength λ_0 of the most unstable vortex-line fluctuations, we can estimate that the collapse creates about $N_s k_0/2\pi$ solitons. This in turn can be used to estimate the size of the solitons after the collapse. We find, using the lattice spacing as the unit of length, that the soliton size after collapse is given by $\xi = \sqrt{2J_V/|V_0|n_0}/\pi$. However, it is clear that the above arguments are only valid if the final soliton size is much smaller than the system size N_s. Otherwise, boundary effects are expected to influence the results considerably. Another way to understand this is to note that in the finite lattice the smallest nonzero wave vector is given by $k_{min} = 2\pi/N_s$. Moreover, we can see from Eq. (5) that the dynamical instability exist only in the regime where $|k| < 2\sqrt{|V_0|n_0/J_V} = k_c$. Therefore, the vortex line in an optical lattice can only be dynamically unstable if $k_{min} \ll k_c$. This implies the condition

$$\frac{\pi}{N_s}\sqrt{\frac{J_V}{|V_0|n_0}} \ll 1, \tag{6}$$

which turns out to be equivalent to the requirement that the final soliton size must be much smaller than the system size.

The above arguments based on a Bogoliubov approach are valid only when the deviations from the initial homogeneous system are small. Given enough time the disturbances in the dynamically unstable system grow large

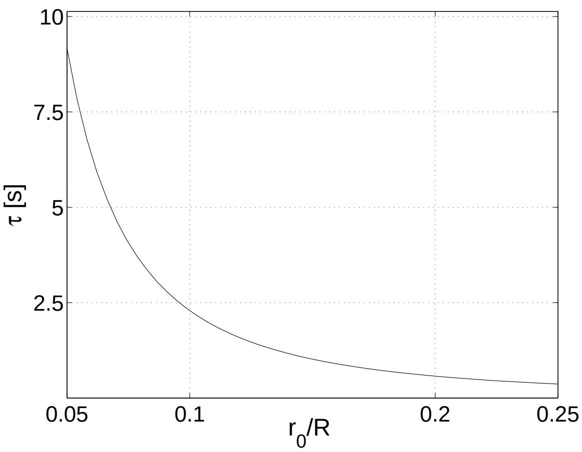

FIG. 2: The timescale for the vortex line collapse as a function of the initial displacement r_0 relative to the condensate size R. For concreteness we further assumed that 1000 ^{87}Rb atoms were trapped in each site with $\Omega = 0.2\,\omega_r$ and $\omega_r = 2\pi\cdot 100\,\text{Hz}$. The lattice depth was $V_L = 15\,E_r$ and the lattice spacing was $d = 398\,\text{nm}$.

and the Bogoliubov approach fails. In this limit we have to solve the discrete time-dependent Gross-Pitaevskii equation numerically. In Fig. 3 we show the time evolution of a displaced vortex line that was initially almost straight. With the parameters used in this figure, which was obtained by a fourth-order Runger-Kutta method, the vortex line is seen to collapse into three bright solitons.

Discussion.— The *in situ* imagining of the three-dimensional vortex line is difficult, but not impossible [16, 19]. However, in many experiments the magnetic trap potential is turned off and the condensate is allowed to expand in order to facilitate the imaging of the Bose-Einstein condensate. In an earlier paper [15] we showed that during the expansion the structure of the three-dimensional vortex line in a good approximation only changes by a scale factor and therefore vortex-line solitons are observable even after the expansion of the Bose-Einstein condensate.

This offers the opportunity to study experimentally a variety of other interesting phenomena. One intriguing possibility is to study a Bose-Einstein condensate with more than one vortex line. In this manner it is possible to study how solitons in different vortex lines interact. The possibility of breather solutions is also an exiting prospect [20].

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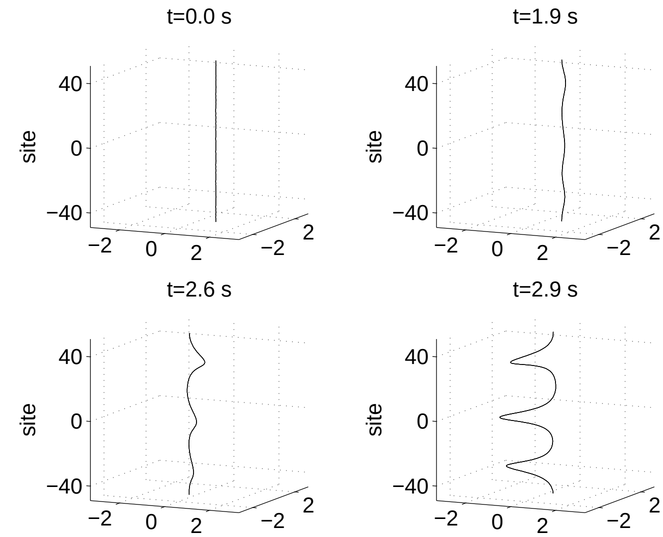

FIG. 3: Time evolution of the vortex line during its collapse into a train of three bright solitons. Initially the vortex line was displaced from the center of the Bose-Einstein condensate by $R/4$. Furthermore, the time evolution was seeded by an unobservable amount of white noise to simulate experimental fluctuations. Each site had 1000 ^{87}Rb atoms, the lattice depth was $V_L = 15\,E_r$, $\Omega = 0.2\,\omega_r$, and $\omega_r = 2\pi\cdot 100\,\text{Hz}$. The lattice spacing was $d = 398\,\text{nm}$. The transverse unit of length is $\sqrt{\hbar/m\omega_r}$.

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